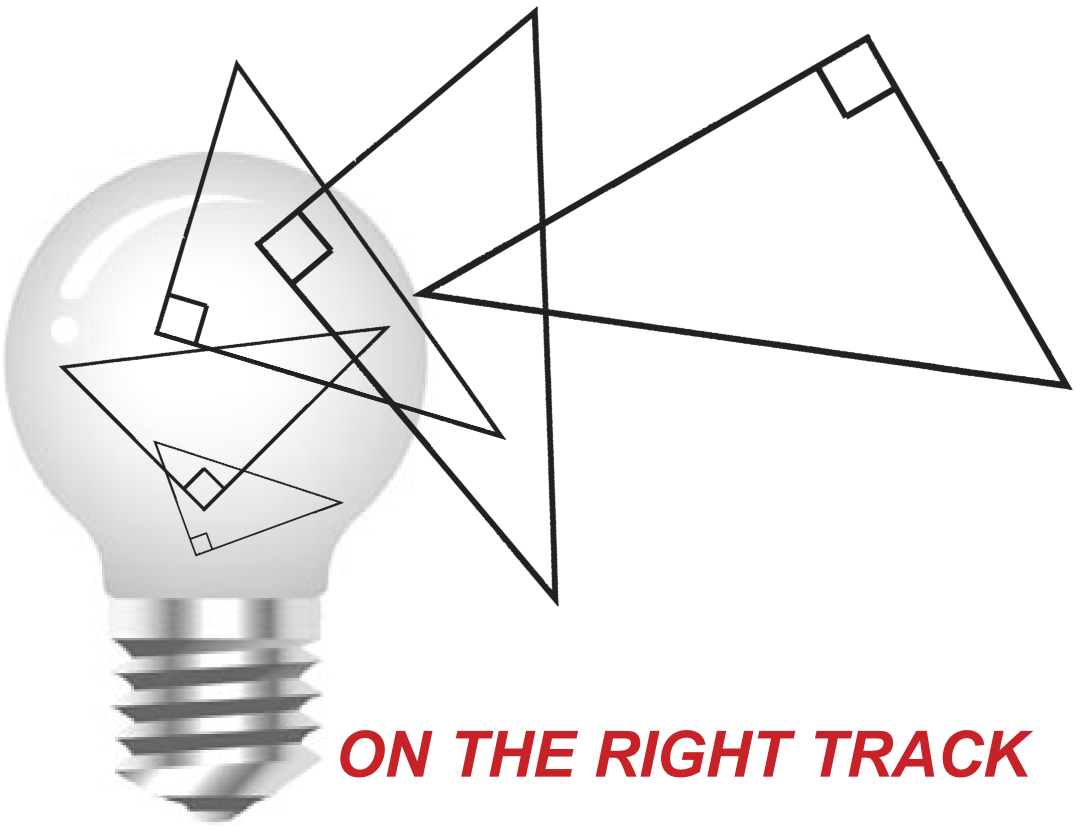
Key Idea 1—Mathematical Reasoning:

Students use mathematical reasoning to analyze mathematical situations, make conjectures, gather evidence, and construct an argument.

Overview:

Mathematical reasoning should be an integral part of the teaching and learning activities in prekindergarten through grade 12 that support each of the other six key ideas. Mathematical reasoning includes such areas as inductive and deductive reasoning, patterns and functions, logic, and various problem-solving strategies. On the elementary and intermediate levels, students exhibit mathematical reasoning when they solve problems, organize, classify, count, and look for patterns. In high school, mathematical reasoning expands to making and proving conjectures, finding examples and counterexamples, applying results, and discovering, explaining, and justifying solutions.



ON THE RIGHT TRACK

Description:

Right triangles and the Pythagorean theorem provide an opportunity for students at all levels to reason mathematically. Through investigations, students discover the relationships between the lengths of the sides of a right triangle.

Elementary Performance Indicators

Students will:

- Use models, facts, and relationships to draw conclusions about mathematics and explain their thinking.
- Use patterns and relationships to analyze mathematical situations.
- Justify their answers and solution processes.
- Use logical reasoning to reach simple conclusions.

PreK – K

1. Give students cutouts of **triangles**. Some of the triangles should have a right angle, some should have an obtuse angle, and some should have only acute angles (see Activity Sheet 1).
2. Have students sort the triangles into piles and explain the various classification systems they used. At some point in the discussion, focus students' attention to the different **angles** of the triangles. Have students explore which angles are bigger than, smaller than, and the same size as others.
3. Have students locate and identify various angles in the classroom.
4. Name and identify a triangle with a right angle. Ask students to find the other triangles with right angles.
5. Have students locate and identify various right angles in the classroom. Students can do this, using the right triangle cutouts.

Grades 1 – 2

1. Give students a map with all streets running north-south or east-west (see Activity Sheet 2). Each street is either parallel or perpendicular to all other streets. Several buildings are located on the map.
2. Have students determine and draw the **shortest** path between two buildings if they must walk along a street. Then have students determine and draw the shortest path between the same two buildings if they can cut across blocks. Ask students what they notice about the triangles they drew. (They are right triangles!)
3. Have students compare distances to discover that the shortest distance between two **points** is a **straight line**. Students may use a **ruler** or a nonstandard **measuring tool** (such as a string).

Grades 3 – 4

1. Give students stirrers cut to lengths of 4, 5, 6, 8, 10, 12, and 13 **centimeters**.
2. Challenge students to build one or more **right triangles** with the stirrers. Have students record the lengths of the stirrers they used.
3. Ask students to explain what they discovered and draw some conclusions about the lengths of the sides of right triangles (e.g., the longest side is opposite the right angle, the length of the longest side is less than the sum of the other two sides).

Intermediate Performance Indicators

Students will:

- Apply a variety of reasoning strategies.
- Make and evaluate conjectures and arguments, using appropriate language.
- Make conclusions based on inductive reasoning.
- Justify conclusions involving simple and compound (i.e., and/or) statements.

Grades 5 – 6

1. Give students **data** on **right triangles** with **integer** side lengths as given in the table below.

Leg	Leg	Hyp			
X	***Y***	***Z***	X^2	Y^2	Z^2
3	4				
5	12				
6	8				
7	24				
8	15				
9	12				

NOTE: Teacher could display the chart using the LIST feature of a graphing calculator.

2. Have students use **graph paper** and a **straightedge** to draw right triangles with legs *X* and *Y* as given in the table. Then, using a strip of graph paper as a measuring device, students should measure and record the length of the corresponding **hypotenuse**, *Z*. Other values for the table (X^2, Y^2, Z^2) should also be calculated and recorded.
3. Ask the students to make conjectures about the relationships among the numbers in the table by **reasoning inductively**.

Grades 7 – 8

1. Have students use graph paper and a ruler to draw right triangles with legs *X* and *Y* as given in the table below. Using the ruler, students should measure and record the length of the corresponding hypotenuse, *Z*. Other values for the table (X^2, Y^2, Z^2, $X^2 + Y^2$) should be calculated and recorded.

Leg	Leg	Hyp				
X	***Y***	***Z***	X^2	Y^2	Z^2	$X^2 + Y^2$
3	4					
5	12					
6	8					
7	24					
8	15					
9	12					

NOTE: A graphing calculator simplifies making the list and allows many more examples to be explored.

2. Have the students make conjectures about the relationship between the lengths of the sides of right triangles by reasoning inductively. Discussion should lead to the Pythagorean relationship.
3. Students can extend their conjectures and inductive reasoning by considering sides that yield a hypotenuse that is not an integer (e.g., 4 and 5).

Commencement Performance Indicators

Students will:

- Construct simple logical arguments.
- Follow and judge the validity of logical arguments.
- Construct proofs based on deductive reasoning.

Math A

1. Have students apply the **Pythagorean theorem** for **right triangles** to derive the **formula** for the **distance** between any two points, (x_1,y_1) and (x_2,y_2), in the **coordinate plane**
   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
   (see Activity Sheet 3).
2. This activity will provide students with an opportunity to make a **valid argument** and to follow and judge the validity of an argument.

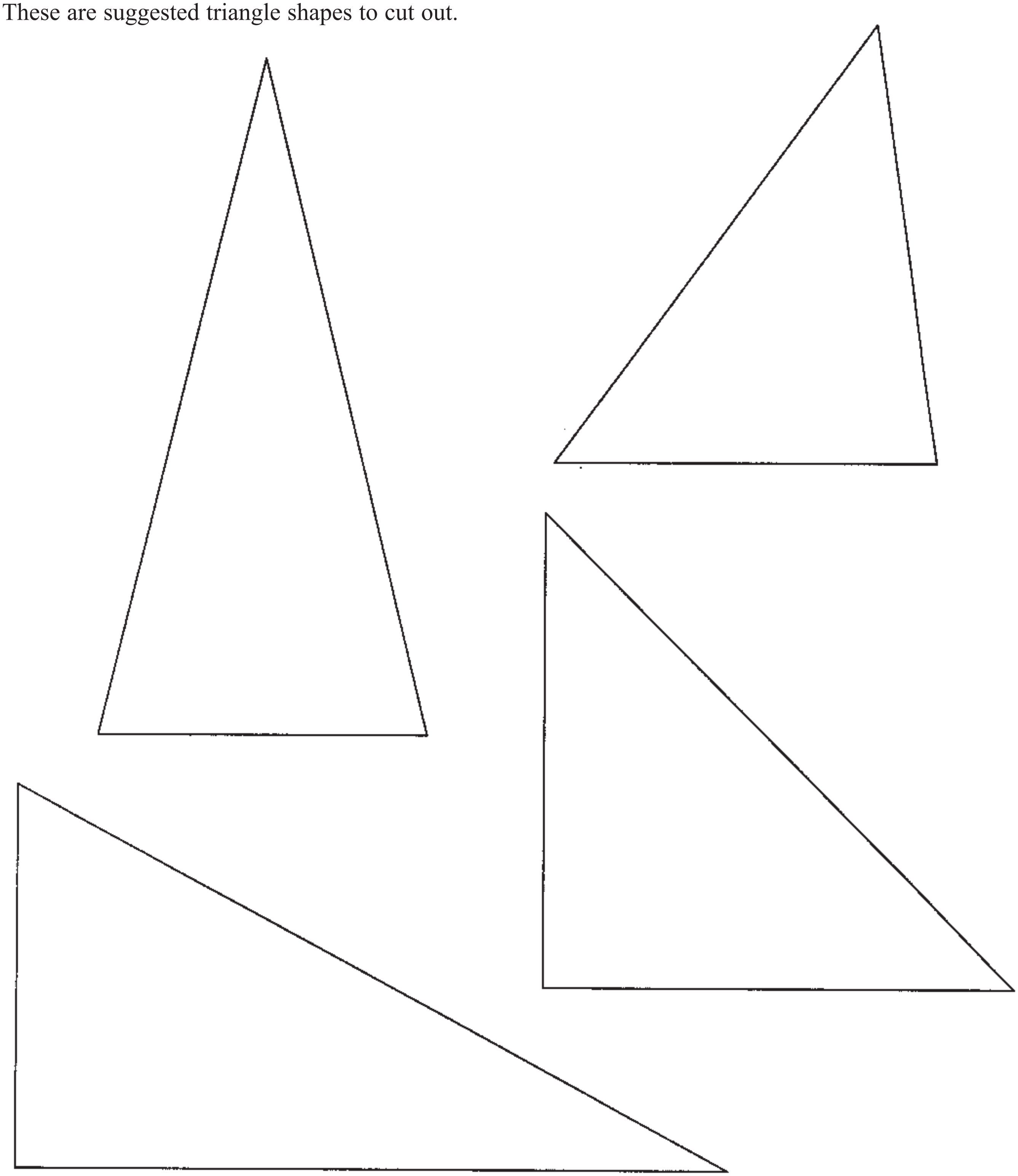
Math B

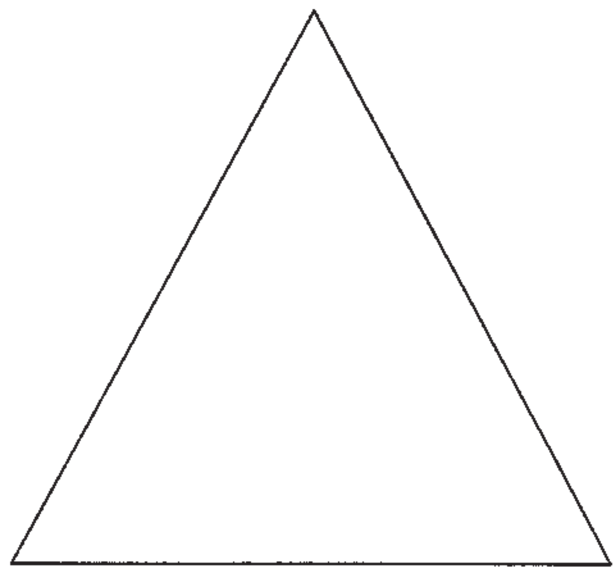
1. Have students prove the Pythagorean theorem for right triangles. (One way to do this is to use relationships that exist from the **similar triangles** formed when an **altitude** is drawn from the **vertex** of the right angle to the hypotenuse.) This activity will provide students with an opportunity to construct a proof based on **deductive reasoning** (see Activity Sheet 4).

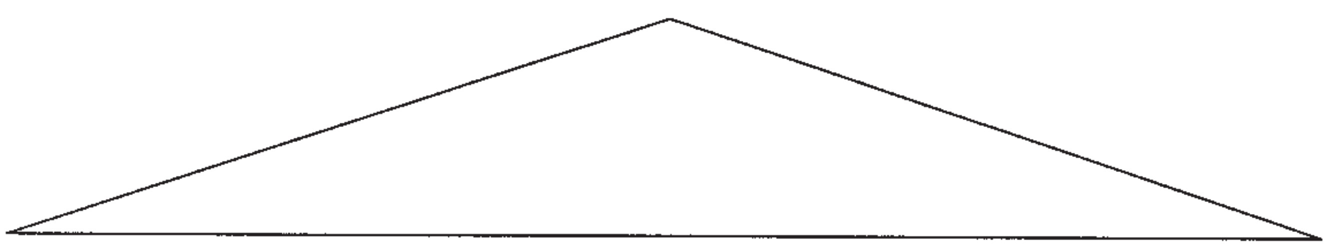
Activity Sheet 1

TRIANGLE SHAPES

These are suggested triangle shapes to cut out.



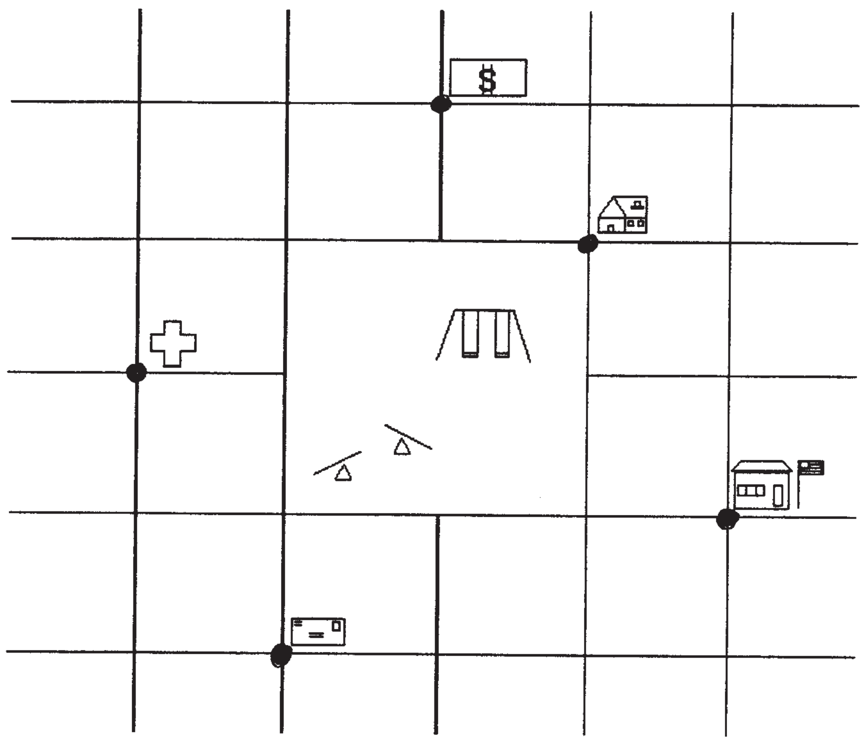




Activity Sheet 2

LET'S TAKE A WALK

CITY MAP



Teacher Notes

Review with the students what each symbol on the map represents; for example, the envelope represents the post office, the dollar bill represents the bank, and the first aid symbol represents the hospital. The students should measure distances from one street corner to another. Tracing their paths on patty paper or tracing paper may help the students compare the distances.

The sample questions below can be used with Activity Sheet 2. (Have students draw the path that they take.)

1. If you can walk only along streets, how far is it from your house to
    a. the hospital?
    b. the post office?
    c. the bank?
    d. the school?

2. If you can cut across blocks, about how far is it from your house to
    a. the hospital?
    b. the post office?
    c. the bank?
    d. the school?

3. If you can walk only along streets, how far is it from the school to
    a. the hospital?
    b. the post office?
    c. the bank?

4. If you can cut across blocks, about how far is it from the school to
    a. the hospital?
    b. the post office?
    c. the bank?

5. What did you notice about the triangles you drew?

6. Did you notice anything about your path if you cut across blocks?

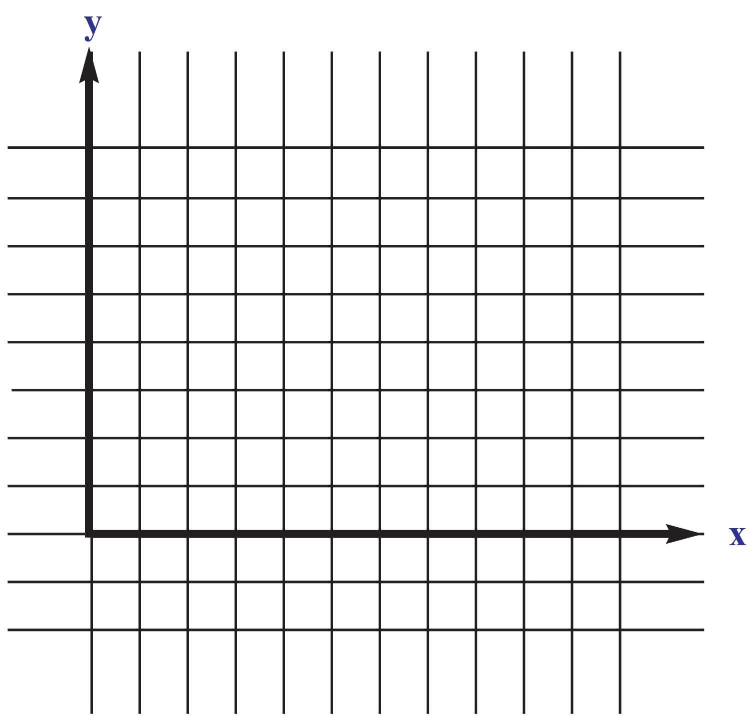
Activity Sheet 3

FINDING THE DISTANCE FORMULA

On the grid below, graph a right triangle in quadrant I. Label the acute angles A and B and the right angle C. The coordinates of A are (x_1,y_1) and the coordinates of B are (x_2,y_2).

Using the Pythagorean theorem, derive the formula to calculate the distance between any two points, A (x_1,y_1) and B (x_2,y_2) in the coordinate plane.

Explain how you derived the formula.

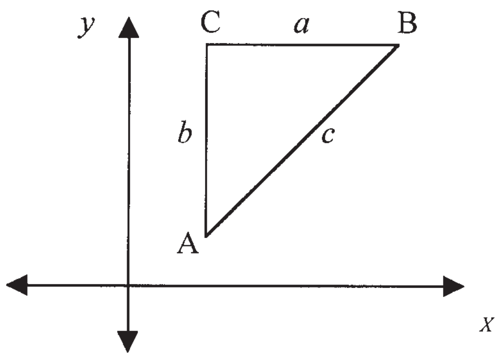


Teacher Notes

Have students apply the Pythagorean theorem to derive the formula for the distance between any two points in the coordinate plane ($d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$). This activity will provide students with an opportunity to make a valid argument and to follow and judge the validity of an argument.

One possible solution is shown below.

The coordinates of A are (x_1, y_1), the coordinates of B are (x_2, y_2), and the coordinates of C are (x_1, y_2).



$a^2 + b^2 = c^2$

$BC = |x_2 - x_1|$ and $AC = |y_2 - y_1|$

$(BC)^2 + (AC)^2 = (AB)^2$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (AB)^2$

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance formula is:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

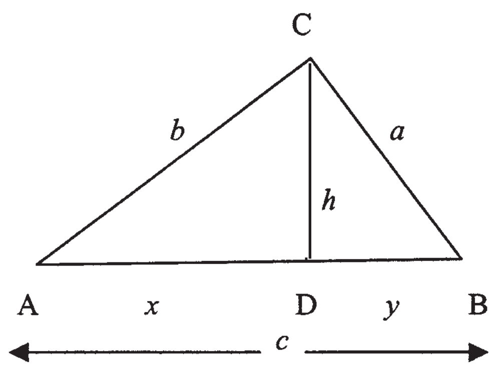
Activity Sheet 4

PYTHAGOREAN THEOREM

Given right triangle ABC with altitude $\overline{CD}$ drawn to hypotenuse $\overline{AB}$, prove the Pythagorean theorem.

Given: Right Δ ABC, right $\angle$ C, legs $\overline{BC}$ (a) and $\overline{AC}$ (b), hypotenuse $\overline{AB}$ (c), altitude $\overline{CD}$ (h)

Prove: $a^2 + b^2 = c^2$



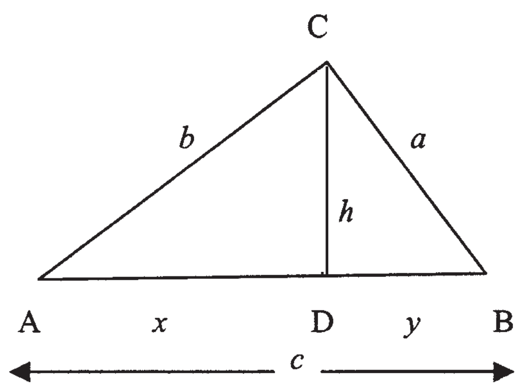
Statements	Reasons
1.	1.

Teacher Notes

Given right triangle ABC with altitude $\overline{CD}$ drawn to hypotenuse $\overline{AB}$, prove the Pythagorean theorem.

Given: Right Δ ABC, right $\angle$ C, legs $\overline{BC}$ (a) and $\overline{AC}$ (b), hypotenuse $\overline{AB}$ (c), altitude $\overline{CD}$ (h)

Prove: $a^2 + b^2 = c^2$



Statements	Reasons
1. Right Δ ABC, right $\angle$ C, altitude $\overline{CD}$ (h)	1. Given.
2. $CD \perp AB$	2. An altitude of a Δ is the line segment drawn from any vertex perpendicular to the opposite side.
3. $\angle$ ADC and $\angle$ BDC are right angles	3. Perpendicular lines form right angles.
4. $\angle ADC \cong \angle BDC \cong \angle ACB$	4. All right angles are congruent.
5. $\angle A \cong \angle A$, $\angle B \cong \angle B$	5. Reflexive property of equality.
6. $\Delta CBD \sim \Delta ABC$ $\Delta ACD \sim \Delta ABC$	6. Two triangles are similar if two angles of one triangle are congruent to two angles of the other triangle.
7. $\frac{c}{a} = \frac{a}{y}$, $\frac{c}{b} = \frac{b}{x}$	7. If two triangles are similar, the corresponding sides are in proportion.
8. $a^2 = cy$ $b^2 = cx$	8. In a proportion, the product of the means equals the product of the extremes.
9. $a^2 + b^2 = cx + cy$	9. Addition property of equality.
10. $c = x + y$	10. The whole is equal to the sum of its parts.
11. $c^2 = cx + cy$	11. Multiplication property of equality.
12. $a^2 + b^2 = c^2$	12. Substitution property.